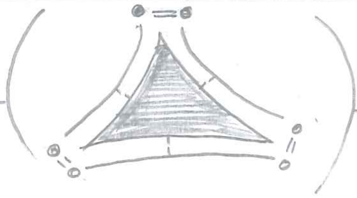
CAT 3

Homology



Quest 0 Find computable homotopy-invariants of simplicial complexes — these are assignments of the form

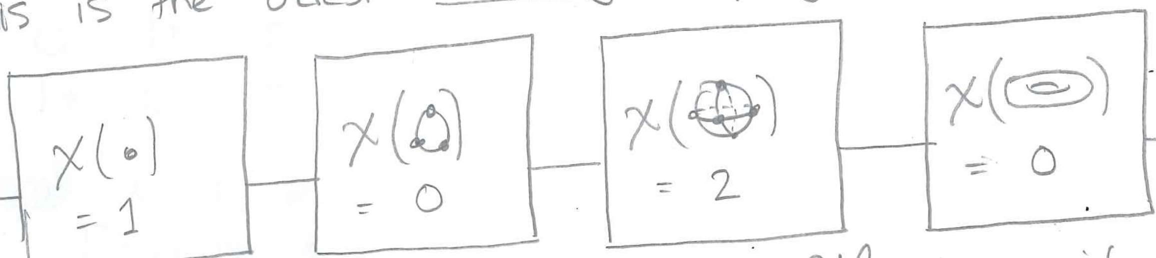
$$\boxed{K \longmapsto I(K)}$$

of algebraic objects $I(K)$ to simplicial complexes K so that if K and L are homotopy-equivalent then $I(K)$ and $I(L)$ are "equal".

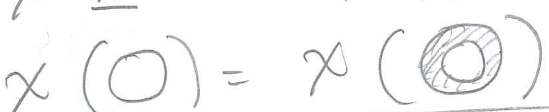
Def 1 The <u>EULER CHARACTERISTIC</u> of a finite simplicial complex K is the alternating simplex count

$$\boxed{\chi(K) = \sum_{i=0}^{\dim K} \#\{\sigma \in K \mid \dim \sigma = i\}}$$

This is the oldest <u>interesting</u> topological invariant.



Note 2 Don't let the simplicity of χ fool you — it conceals several important and fascinating properties, as we will see later in this course. For one thing, χ <u>is</u> a homotopy invariant, eg



Eg 3
- A number $k \in \mathbb{Z}_{\geq 0}$ is called <u>SQUAREFREE</u> if $\nexists$ prime p so that p^2 divides k.
- For each such k, let $\pi(k) = \{\text{list of primes dividing } k\}$. Note that this list is closed under taking subsets.
- Now $\forall\, n \in \mathbb{Z}_{\geq 0}$ define $K_n = \{\pi(k) \mid k \leq n \text{ is squarefree}\}$

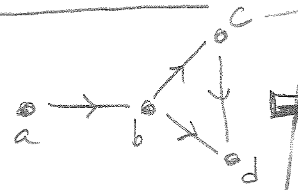
- This forms a simplicial complex [the i-dim'l simplices are lists of length $= i-1$].
- So now we have a sequence of Euler characteristics: $\chi(\underline{K}_1), \chi(\underline{K}_2), \chi(\underline{K}_3), \ldots$ one for each n.

Thm [A Björner 2011]

$$\boxed{\chi(\underline{K}_n) \in O(n^{1/2+\varepsilon}) \quad \forall \varepsilon > 0} \iff \boxed{\text{Riemann Hypothesis.}}$$

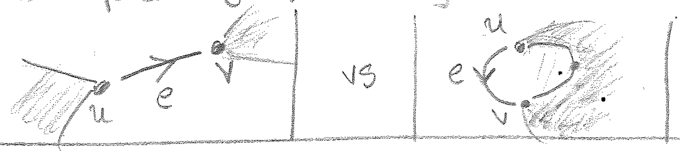
CHAIN COMPLEXES

Note 4 Directed graphs have "incidence matrices" which record connections between vertices and edges:

Eg: $G =$  $\Rightarrow I_G =$

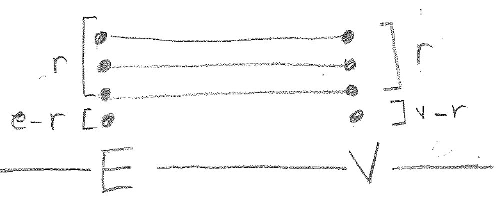
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} ab & bc & bd & cd \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 \\ +1 & -1 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ 0 & 0 & +1 & +1 \end{bmatrix} \end{array}$$

What does this MEAN? Well, this is a <u>linear map</u> [vector space of edges] $\longrightarrow$ [vector space of vertices] that sends each $(u \xrightarrow{e} v)$ to $\underbrace{v-u}_{\text{"algebraic boundary"}}$..



Prop 5 Let G be a finite directed graph with v vertices, e edges and incidence matrix I_G of rank r. Then,

$$\boxed{\begin{aligned} \#[\text{components of } G] &= v - r \\ \#[\text{loops in } G] &= e - r \end{aligned}}$$

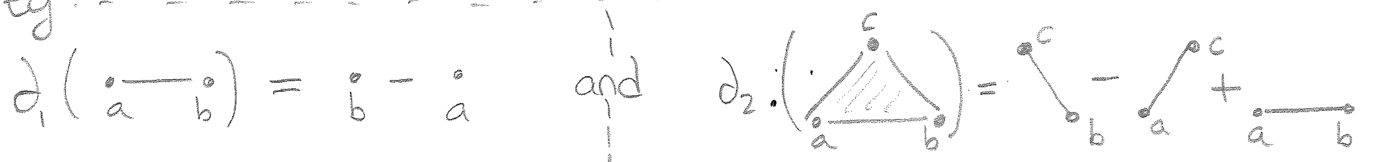


Def 6 a) By an "<u>orientation</u>" of a simplicial complex K we mean an ordering of all the vertices, i.e., an injective map $K_0 \longrightarrow \mathbb{N}$. [This is the simplicial version of a directed graph]. We always write $\sigma = (v_0, \ldots, v_n)$ in <u>order</u>.

b) Let K be an oriented simplicial complex. The ~~ORDERED!!~~ **ALGEBRAIC BOUNDARY** of an n-simplex $\sigma = (v_0, \dots, v_n)$ is the following linear combination of $(n-1)$ simplices:

$$\boxed{\partial_n \sigma = \sum_{i=0}^{n} (-1)^i (v_0, \dots, \cancel{v_i}, \dots, v_n)}$$

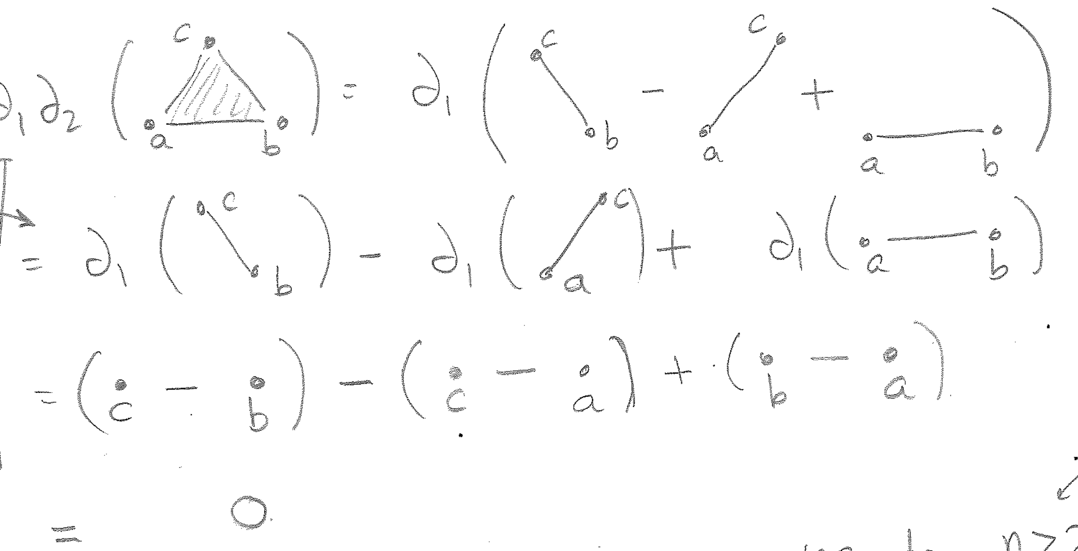
Eg:



Prop 7. For any n-simplex $\sigma = (v_0, \dots, v_n)$ in an oriented simplicial complex K, we have

$$\boxed{\partial_{n-1} \circ \partial_n (\sigma) = 0}$$

Pf Try $n=2$; now $\sigma = (a,b,c)$. We get



Here we have extended ∂_1 from simplices to linear combinations by linearity ...

Problem Sheet 2

Now we have to extend this reasoning to $n > 2$ ... everything occurs twice, but with opposite signs!

Note 8 So if you hand me an oriented simplicial complex of top dimension n, I can build a sequence of vector spaces and linear maps of the form

$$\boxed{C_n^K \xrightarrow{\partial_n^K} C_{n-1}^K \xrightarrow{\partial_{n-1}^K} \cdots \longrightarrow C_2^K \xrightarrow{\partial_2^K} C_1^K \xrightarrow{\partial_1^K} C_0^K}$$

with the property that composing any adjacent pair $\partial_{i-1} \circ \partial_i$ of these maps yields zero, $C_i \xrightarrow{0} C_{i-2}$.

Here C_i^k is the vector space generated by all the oriented i-simplices. To make this precise, we must choose COEFFICIENTS, eg $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}/p$, to fully define scalar multiplication beyond ± 1.

Let's use $\mathbb{Q}$ coeffs here

Def 9 a) A <u>CHAIN COMPLEX</u> is a sequence of vector spaces and linear maps of the form:

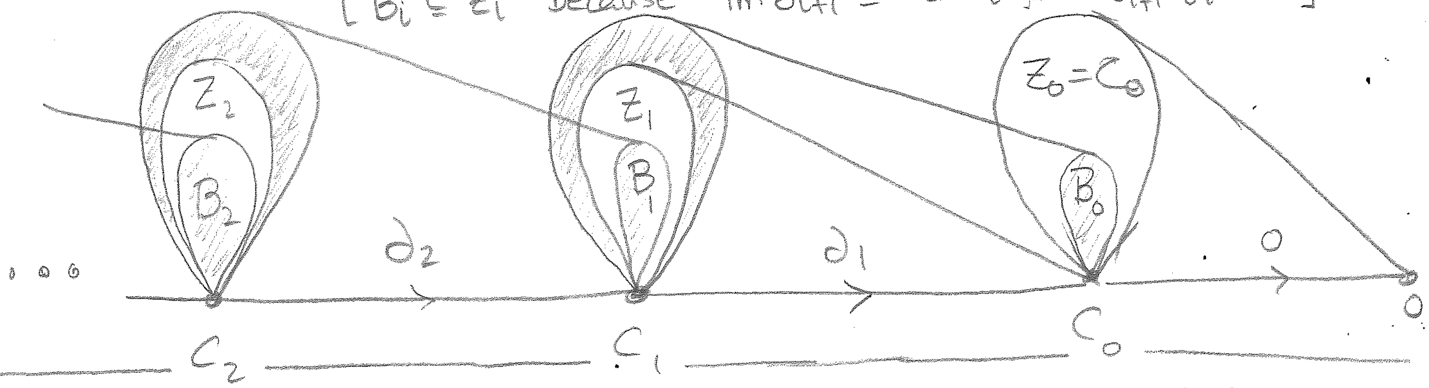
C_i = space of i-chains

$$\text{(this can keep going)} \cdots \boxed{\xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0} \to 0$$

so that $\partial_{i-1} \circ \partial_i : C_i \to C_{i-2}$ is the zero map for all $i > 0$. [Write this as $(C_\bullet, \partial_\bullet)$]

b) The vector space of <u>i-CYCLES</u> is the subspace $Z_i \subseteq C_i$ given by $\ker \partial_i$. And the vector space of i-<u>BOUNDARIES</u> $B_i \subseteq C_i$ is the image $\operatorname{im} \partial_{i+1}$.

[$B_i \subseteq Z_i$ because $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i \iff \partial_{i+1} \circ \partial_i = 0$]



Def 10 The <u>HOMOLOGY GROUPS</u> of a chain complex $(C_\bullet, \partial_\bullet)$ are defined to be the quotients

$$\boxed{H_i = Z_i / B_i}$$

of cycle-spaces by boundary spaces. [So H_i is a SUB-QUOTIENT of C_i, i.e.: the quotient of a subspace]

Note 11 The HOMOLOGY OF A SIMPLICIAL COMPLEX K is defined to be the homology of its chain complex $(C_\bullet^K, \partial_\bullet^K)$ [from Note 8]. It is denoted $H_i(K)$, or when we want to emphasize the chosen field of coefficients $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{Z}/p, \dots\}$, we write it as $H_i(K; \mathbb{F})$.

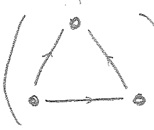
Eg 12

a)

$$H_i(\text{point}; \mathbb{F}) = \begin{cases} \mathbb{F}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

The chain complex is $\left[\to \underset{i}{0} \to \cdots \to \underset{1}{0} \to \underset{0}{\mathbb{F}} \to 0 \right]$

b)

$H_i\Big($  $\Big) = \begin{cases} \mathbb{F}, & i = 0 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$

The chain complex is $\left[\to 0 \to \cdots \to \mathbb{F}^3 \xrightarrow{\alpha} \mathbb{F}^3 \to 0 \right]$ where α is a 3×3 matrix of rank 2.

Facts 13

$\boxed{A}$ Homology is a <u>FUNCTOR</u>: not only does it attach vector spaces $H_i(K; \mathbb{F})$ to simplicial complexes K, but it also assigns linear maps $H_i f: H_i(K) \to H_i(L)$ to simplicial maps $f: K \to L$. (More on this later)

$\boxed{B}$ Homology is a homotopy-invariant. So, if K and L are homotopy-equivalent then $H_i(K; \mathbb{F})$ and $H_i(L; \mathbb{F})$ are isomorphic (regardless of i and $\mathbb{F}$)

$\boxed{C}$ Computing Homology of finite simplicial complexes is possible via standard Gaussian elimination of matrix representations of ∂_n^K's.

$\boxed{D}$ Euler characteristic inherits its homotopy invariance from homology: $\boxed{\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \dim H_i(K; \mathbb{F})}$

More on $\boxed{C}$omputations of Homology...

It is important to note that over any FIELD, the computation of homology reduces to computing ranks of all the ∂_i's. [Things are more complicated over $\mathbb{Z}$].

Prop 14 Let K be a simplicial complex (finite, of course!) If $n_i = \#\{i\text{-simplices in } K\}$ is $\dim C_i^K$, and if $r_i = \operatorname{rank}(\partial_i^K)$, then:

$$\dim H_i(K) = n_i - (r_i + r_{i+1})$$

[Note $\dim Z_i = n_i - r_i$ and $\dim B_i = r_{i+1}$]

Note 15

a) Really, we are DIAGONALIZING all the ∂_i's. As we have assumed field coefficients, this means for each ∂_i there are INVERTIBLE MATRICES P_i and Q_i so that

$$\boxed{\partial_i = \underset{(n_i \times n_i)}{P_i} \left[\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r_i} & 0 \\ \hline 0 & 0 \end{array}\right] \underset{(n_{i+1} \times n_{i+1})}{Q_i}}$$

This diagonal matrix is called the SMITH NORMAL FORM of ∂_i, and it looks more interesting over $\mathbb{Z}$-coefficients

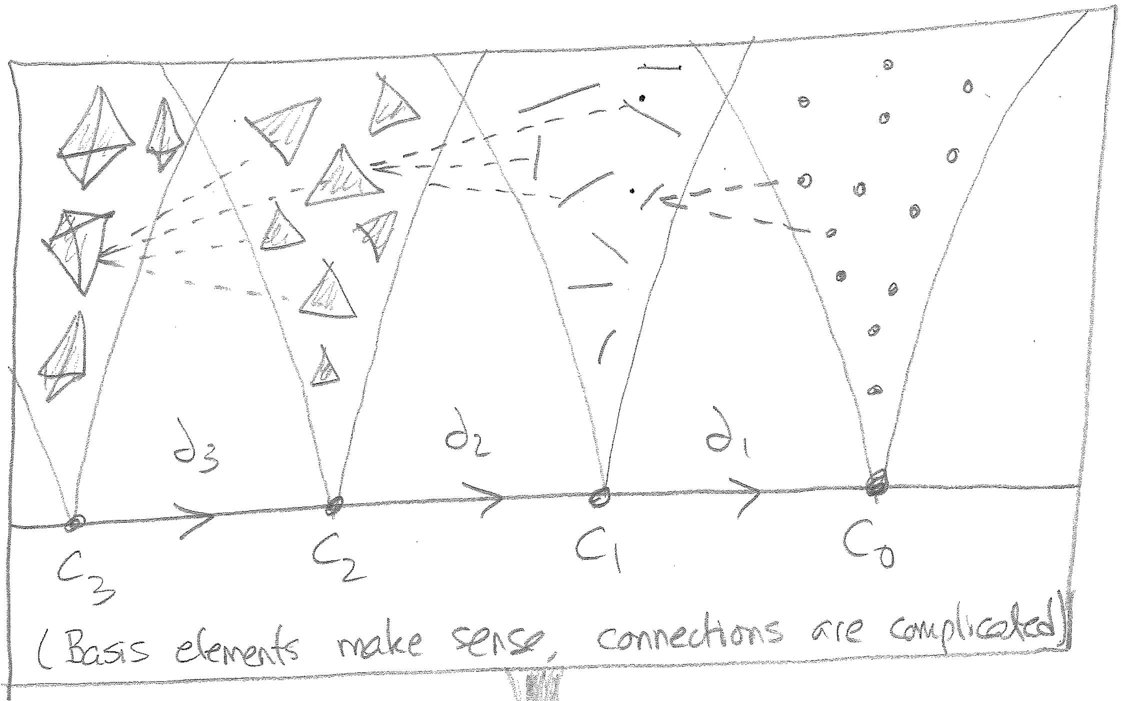
b) Algorithmically, going from ∂_i to its SNF "just" amounts to using row operations to get pivot columns [this produces a change-of-basis matrix P_i] and then column operations to get pivot rows [this produces Q_i]. You can do both row and column ops at once, of course but...

$$\begin{bmatrix} * & * & * & * & * \\ * & * & \partial_i & * & * \\ * & * & & * & * \\ * & * & * & * & * \end{bmatrix} \xrightarrow{P_i \cdot [\partial_i]} \begin{bmatrix} 1 & * & 0 & \cdots & * \\ 0 & 0 & 1 & * \cdots & * \\ \vdots & & \vdots & & \\ 0 & \cdots & 0 & & \end{bmatrix} \xrightarrow{P_i [\partial_i] Q_i} \overset{\text{SNF!}}{\left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]}$$

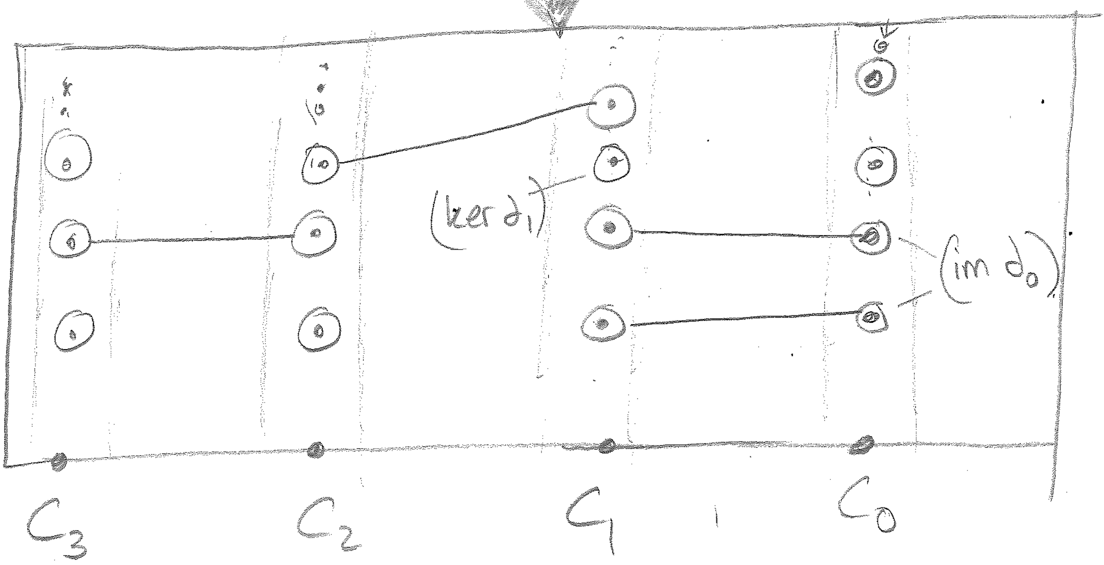
Pic 16

There's a <u>SINGLE PICTURE</u> to keep in mind when going from chain complexes to Homology groups via matrix diagonalization. It often gets lost in the algebraic noise, but I hope you will remember.

DIAGONALIZING $[\partial_i]$ TURNS THIS:



INTO THIS



Here each ⊙ is a weird linear combination of i-simplices, the number of "lines" from C_i to C_{i-1} is the rank of ∂_i, and the UN-LINED ⊙'s count dim H_i.